Exercise 33

Verify that another possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ in Example 4 is $\delta = \min\{2, \varepsilon/8\}$.

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad 0 < |x-3| < \delta \qquad \text{then} \qquad |x^2 - 9| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x-3|$.

$$|x^2 - 9| < \varepsilon$$

$$|(x+3)(x-3)| < \varepsilon$$

$$|x+3||x-3| < \varepsilon$$

On an interval centered at $x = 3$, a positive constant C can be chosen so that $|x+3| < C$.

$$C|x-3| < \varepsilon$$

$$|x-3| < \frac{\varepsilon}{C}$$

To determine C, suppose that x is within a distance 2 from 3.

$$|x-3| < 2$$

$$-2 < x - 3 < 2$$

$$4 < x + 3 < 8$$

$$|x+3| < 8$$

The constant C is then 8. Choose δ to be whichever is smaller between 2 and $\varepsilon/8$: $\delta = \min\{2, \varepsilon/8\}$. Now, assuming that $|x-3| < \delta$,

$$\begin{aligned} |x^2 - 9| &= |(x+3)(x-3)| \\ &= |x+3||x-3| \\ &< 8\left(\frac{\varepsilon}{8}\right) = \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 3} x^2 = 9.$$